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Toward a Formal Proof of the Ke-pler Conjecture

December 17, 2004
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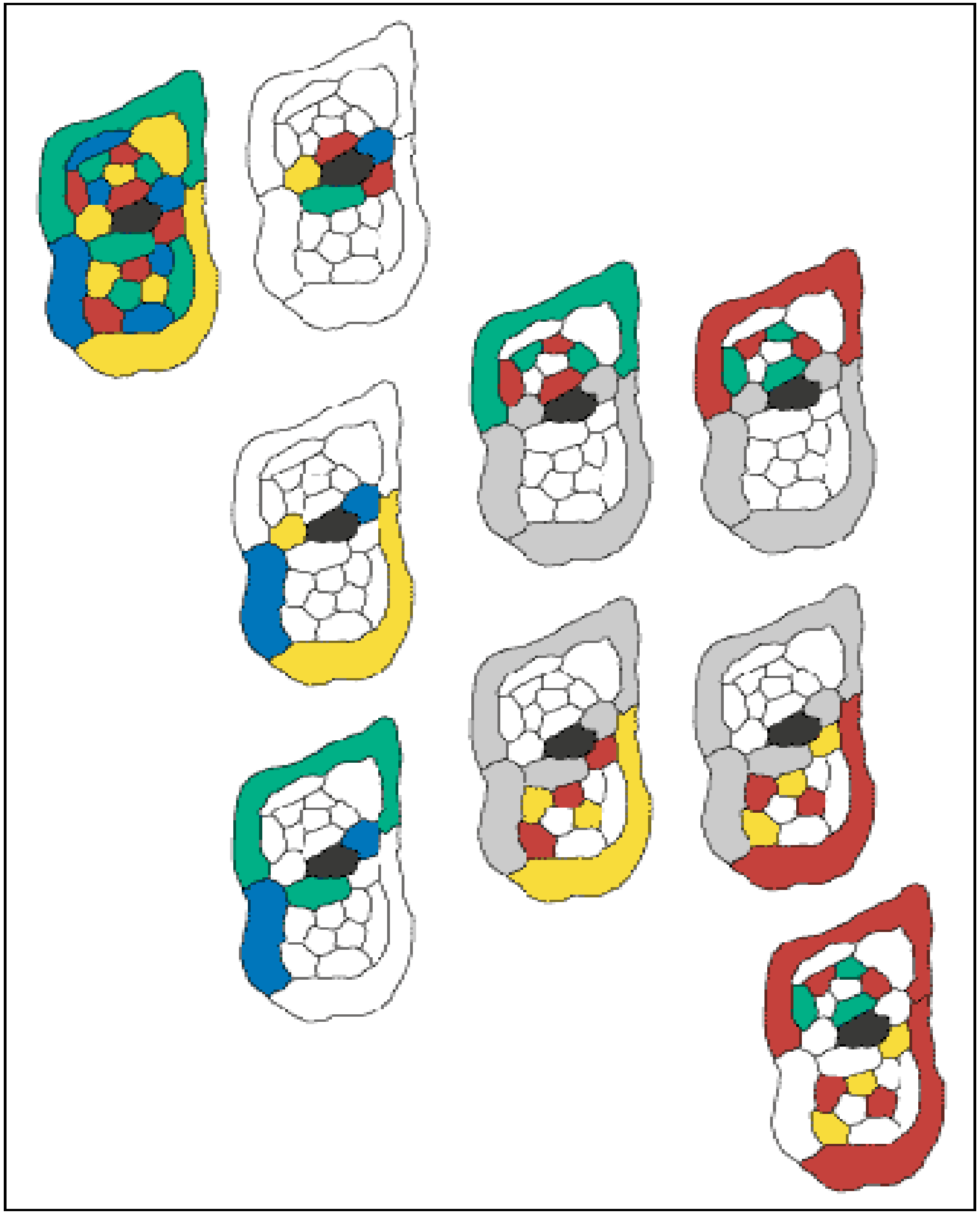
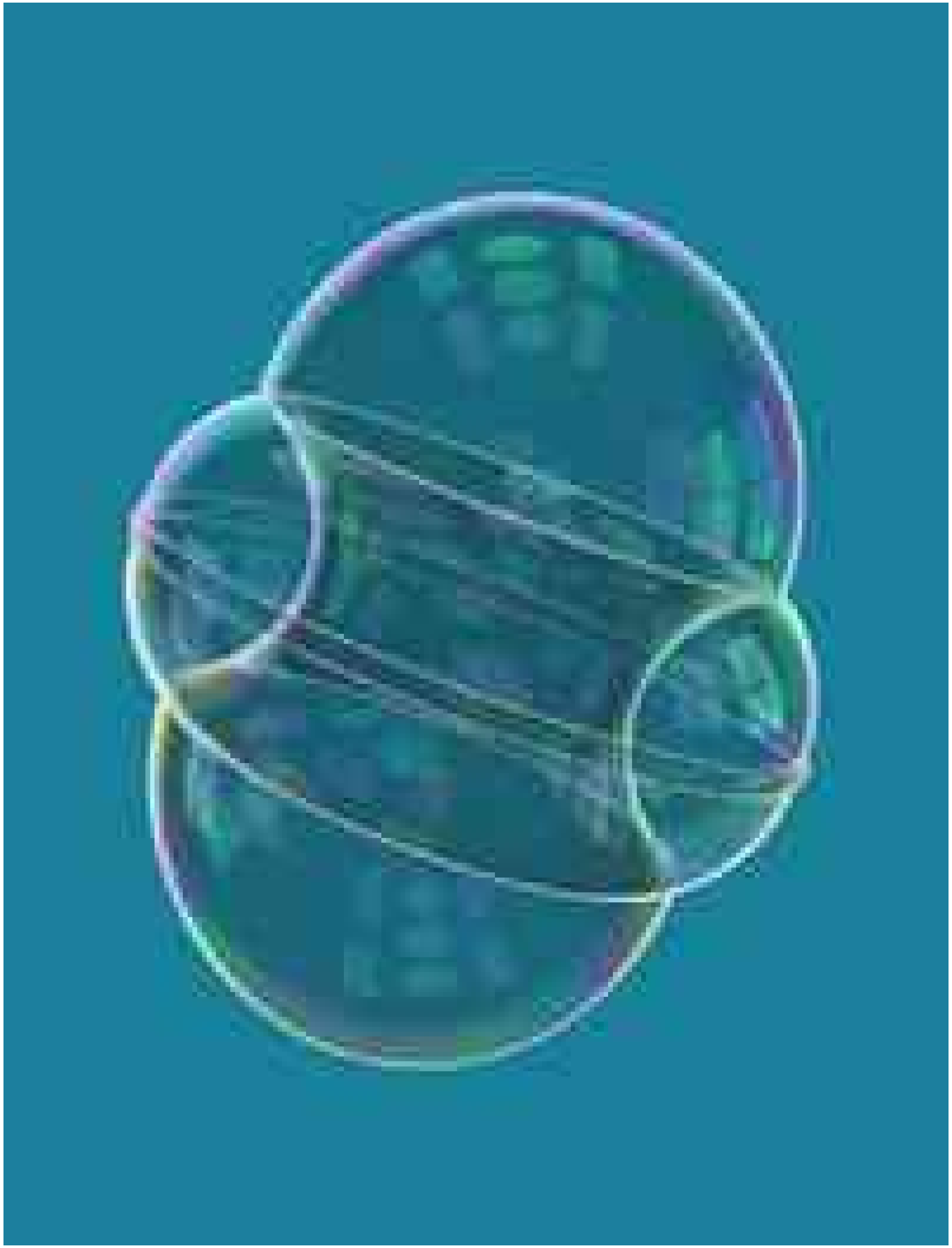


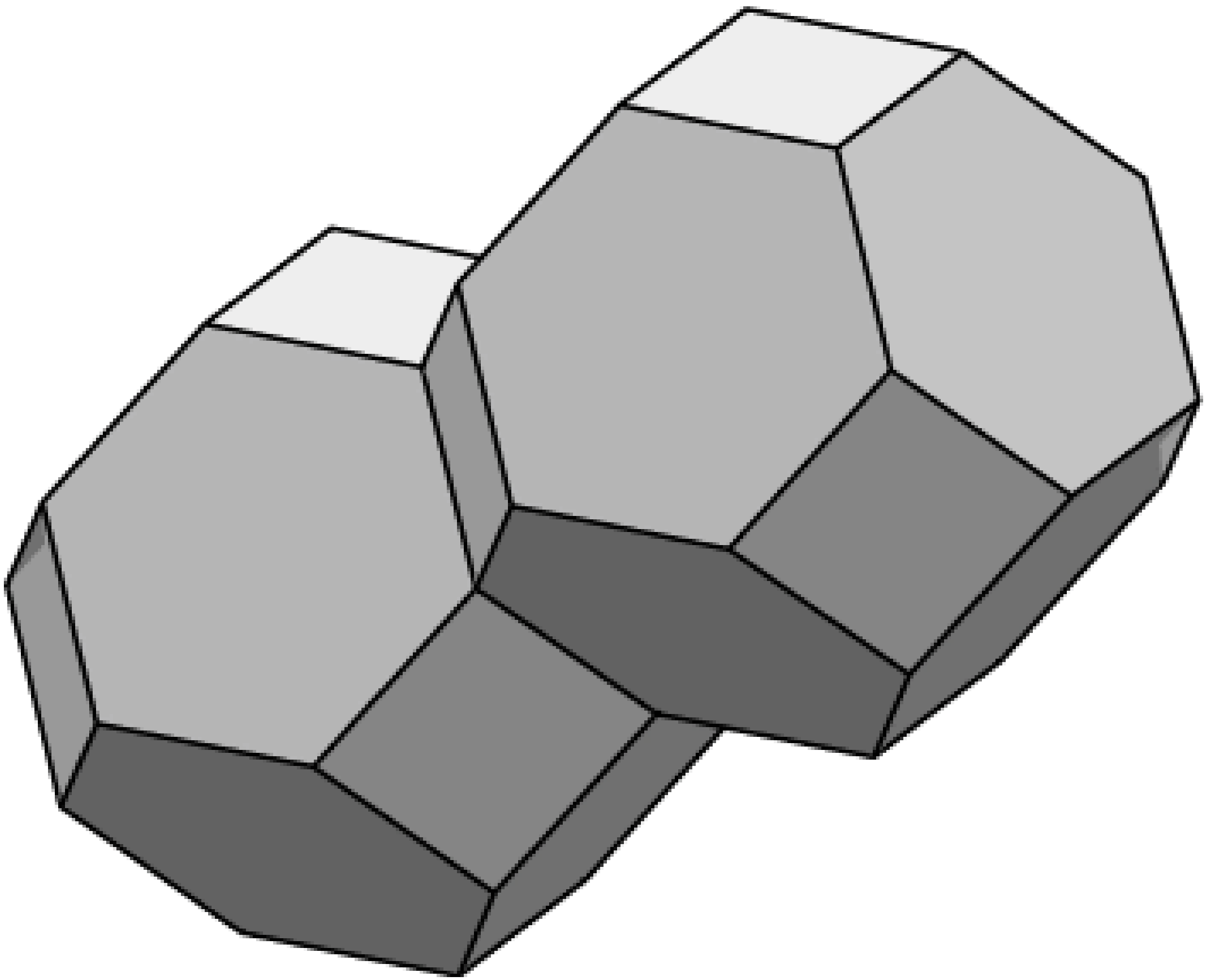


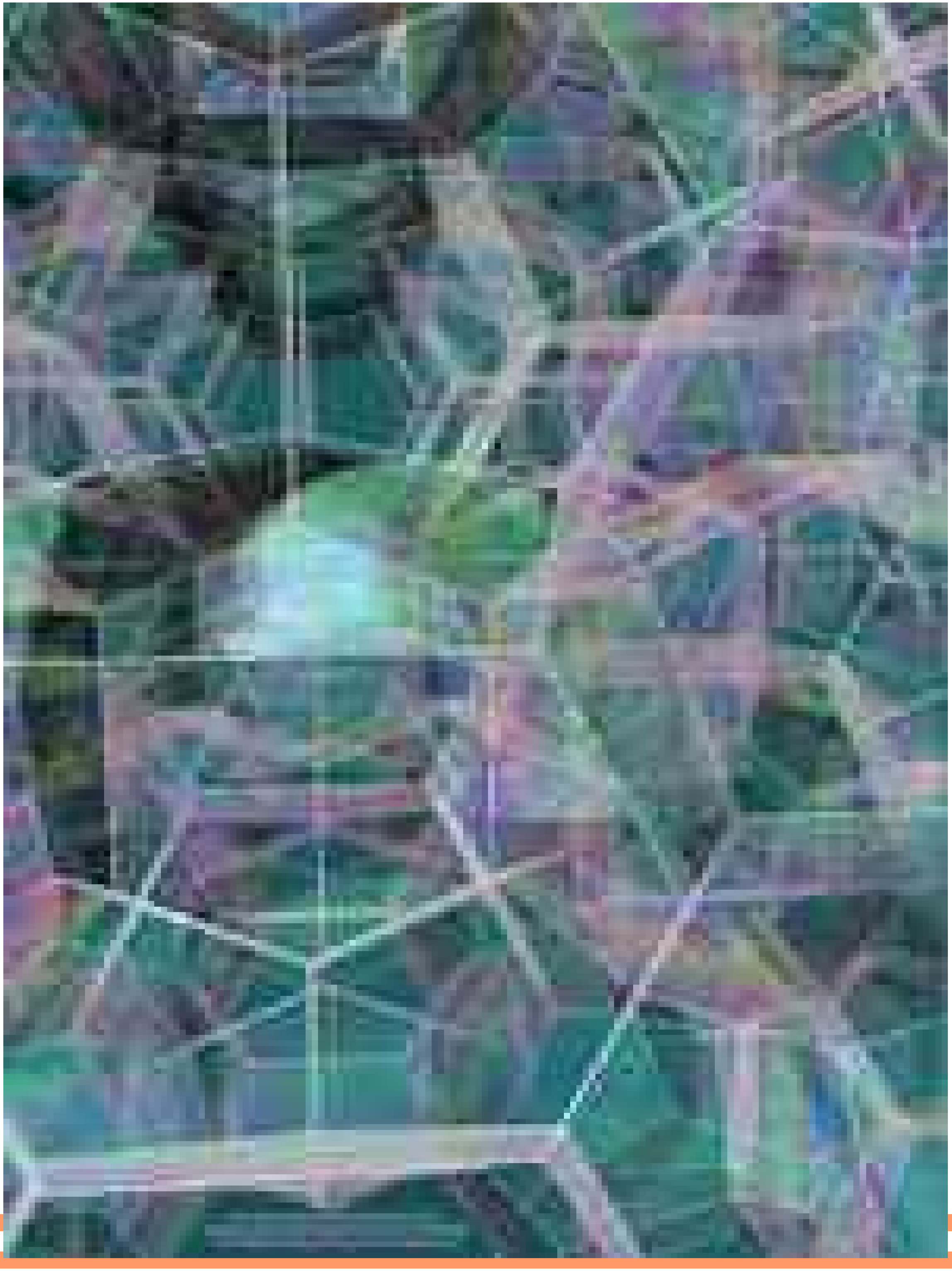
Double Bubble Conjecture

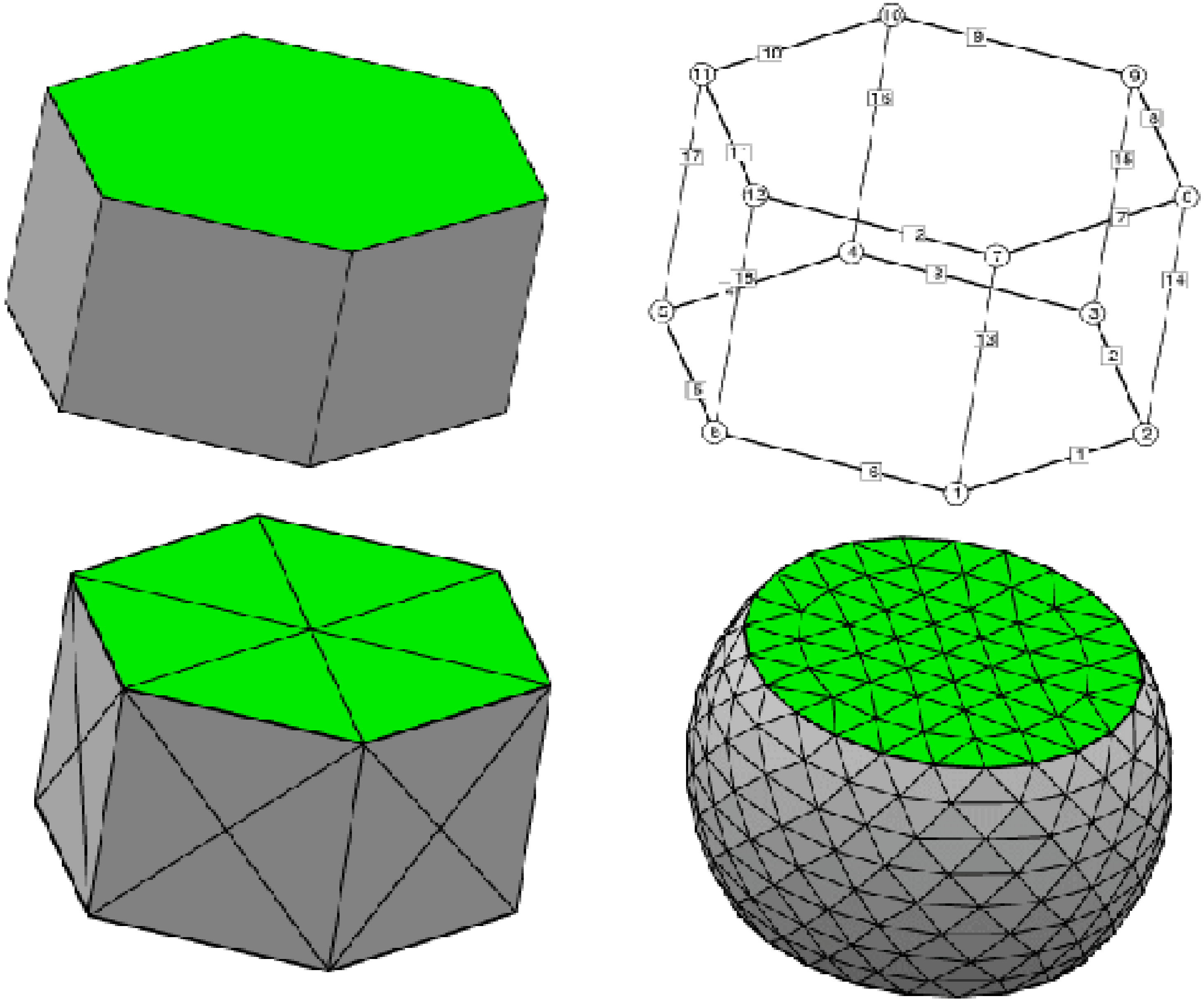
- What is the surface minimizing way of enclosing two equal volumes?
- In 1911, David Boys in his book on Soap Bubbles described the appearance of two bubbles.
- Proved by Hass, Hutchings, and Schlafly in 1995.
- The proof is a mixture of computer calculations and traditional paper-and-pencil mathematics.
- The computer calculations use interval arithmetic to rule out
- other possible configurations.

Proof of the Double Bubble Conjecture

- The proof relies on computer calculations that show that torus bubbles are
  - Not minimal surfaces, or
  - Have area greater than the standard double bubble.
- Calculations used IEEE 754 standard floating point.
- The code takes about 10 seconds to run on a fast (1999) PC and examines about 15K rectangles.

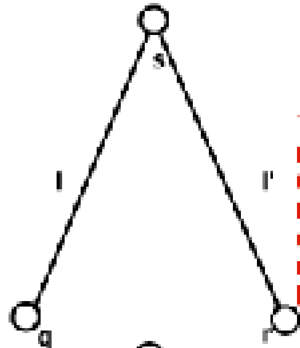




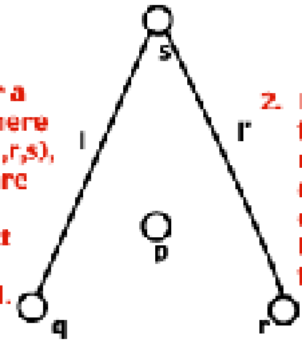


Kelvin Problem

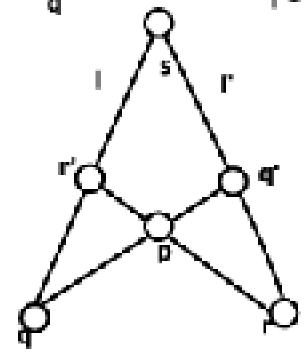
- Find the surface area minimizing partition of space into equal volume cells.
- A counterexample was constructed by Phelan and Weaire using an educated guess and Brakke's "surface evolver software."
- The output from the computer software was later rigorously checked by John Sullivan.



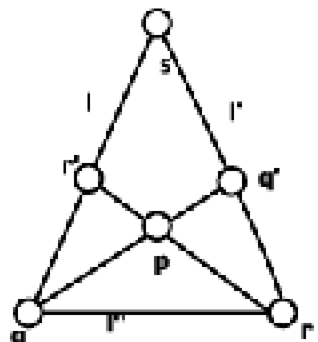
1. By the axioms for a projective plane, there are four points (p,q,r,s), no three of which are colinear. So there must be two distinct intersecting lines l and l' as pictured.



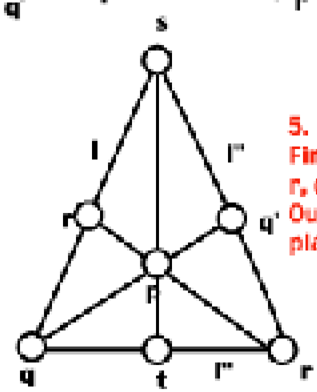
2. Let q and r be the points that lie on l and l', respectively, that are distinct from the point of intersection s, and let p be the fourth point that lies on neither l nor l'.



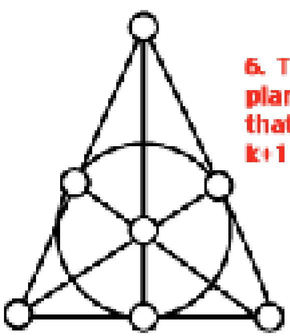
3. Now any two points determine a unique line, and any two lines intersect at a unique point. So create a line from l to l' from q through p. In addition, create another line from l' to l from r through p. We will then have the intersection points q' and r' as illustrated.



4. Now, by the axioms for a projective plane, all points must be connected, thus, we connect q and r, creating line l''. Note that for any two lines (not through p), we must be able to go from a given point on any line, through p, to some point on another given line. Thus, we must create a line from s through p, to l''.



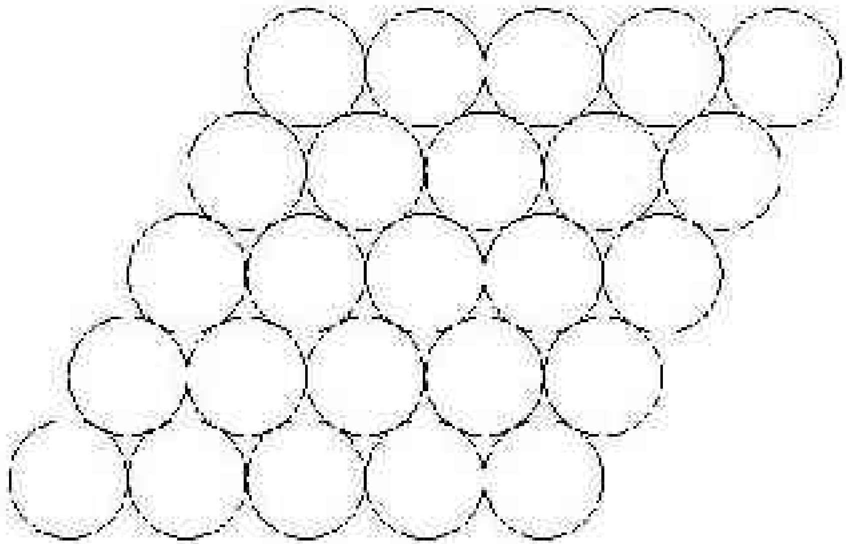
5. We will call our new point t. Finally, we musthave our points r, q' and t connected by a line. Our result is the Fano plane!



6. The Fano plane, and note that $k+1 = 3$.

Nonexistence of the projective plane of order 10

- In 1989, Lam, Thiel, and Swiercz proved the non-existence.
- The calculation relied on an estimated 2000 hours of CRAY computations.
- “Because of the use of a computer, one should not consider these results as a `proof` in the traditional sense,... They are experimental results and there is always a possibility of mistakes.”



Lagrange's Theorem

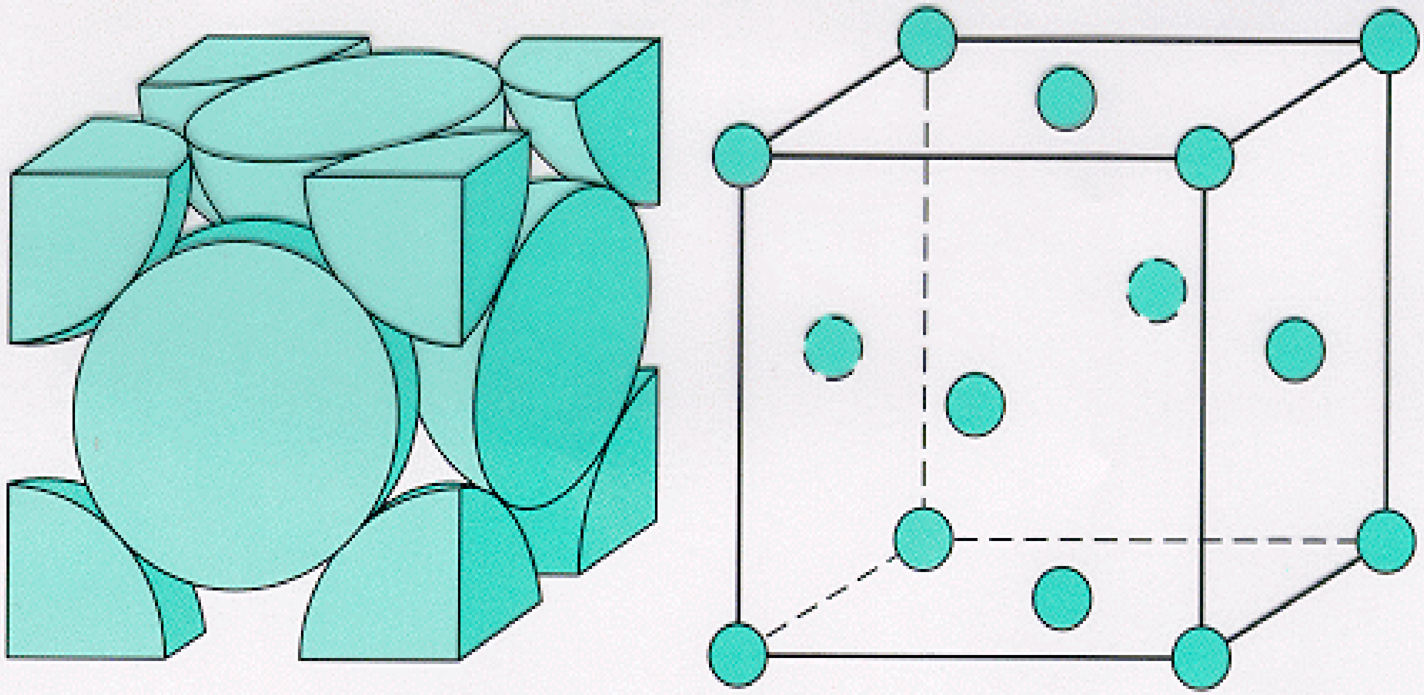
- In 1773, Lagrange proved that the best lattice packing of equal disks is the obvious solution.
- The general 2D packing problem was not settled until around 1900.
- The best lattices were found in dimensions
  - 3 by Gauss in 1836
  - 4, 5, 6, 7, 8 Korkine and Zolotareff, Blichfeldt (1935)
- The best lattice was found in dimension
  - 8, 24 by H. Cohn and A. Kumar in 2004.

Cohn and Kumar's proof of the densest lattice packing in 24 dimensions

- The proof relies on computer calculations performed in PARI.
- The calculations use high-precision floating point arithmetic (3000 digits).
- The calculations take about 6 hours on a PC.

Hyperbolic manifolds

- D. Gabai, Meyerhoff, and N. Thurston (2003) proved that homotopy hyperbolic 3-manifolds are hyperbolic.
- The proof relies on long computer calculations (about 60 CPU days).
- The calculations use IEEE-754. They do not use interval arithmetic, rather they make error-estimates in the paper that become part of the computer code.
- "We pose the simple question: why should one have confidence in our proof?"



(a) (b)

The Kepler Conjecture

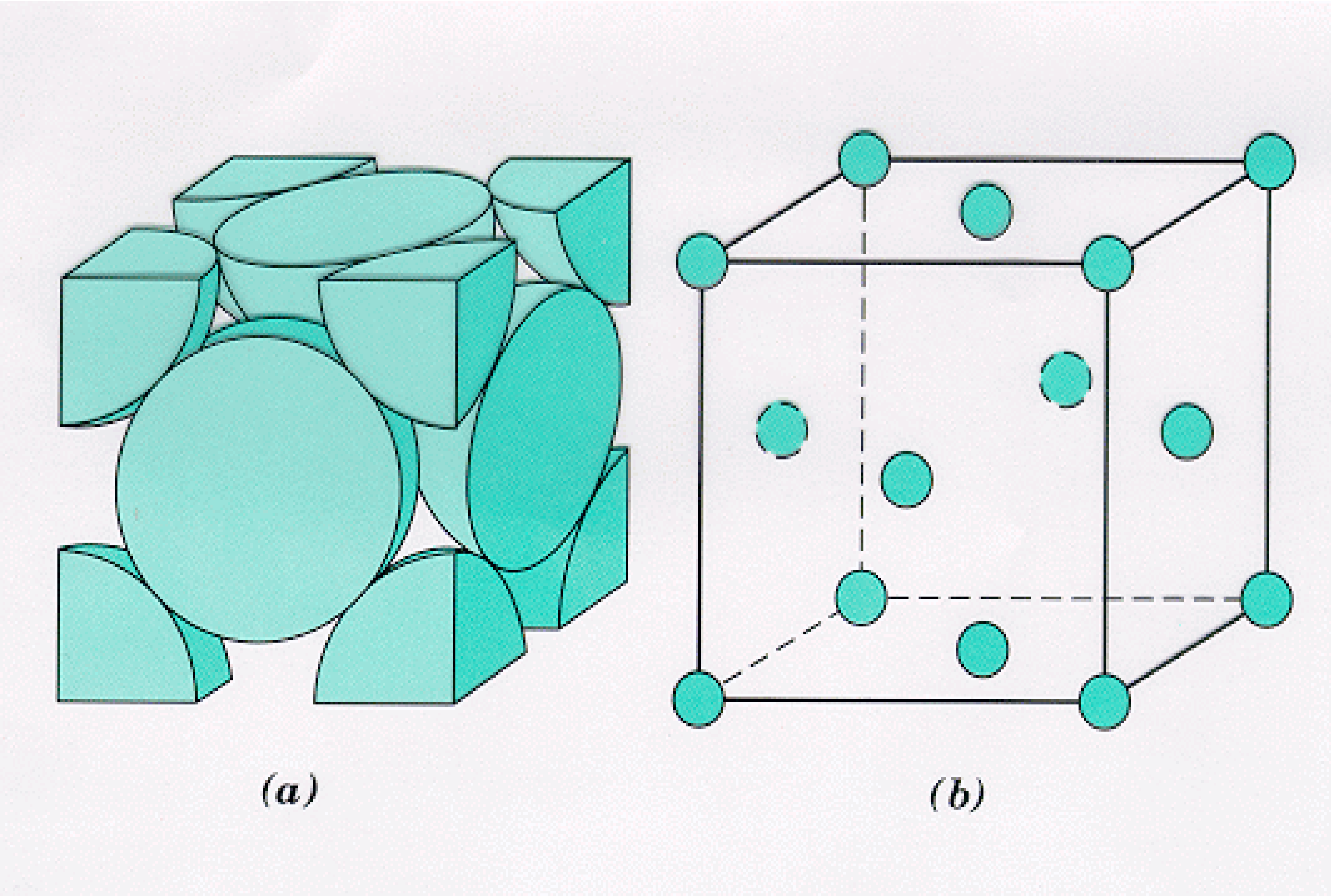
- (Kepler 1610) The cannonball packing is (one of) the densest packings of sphere in 3D.
- The problem was solved in 1998 by com-puters.
  - Interval arithmetic ~3 months on a PC.
  - Linear programming ~1 week on a PC.
  - Planar graph combinatorics, a few hours.
- The FlysPecK project = Formal Proof of the Kepler conjecture

Referees:

- 12 referees were assigned to the paper.
- After 6 months the referees were 99% cer-tain of the proof.
- After 5 more years of checking, the referees were 99% certain of the proof.

The official Annals of Math policy on computers (2004)

- The human part of the proof ... will be refereed for correct-ness in the traditional manner.
- The computer part will be examined for the methods by which the authors have eliminated or minimized possible sources of error: (e.g. round-off error eliminated by interval arithmetic,...).
- [A] second and independent implementation of a computer assisted proof has its place, and should be properly recog-nized just as a duplication of an experiment is recognized in physics.


(a)
(b)

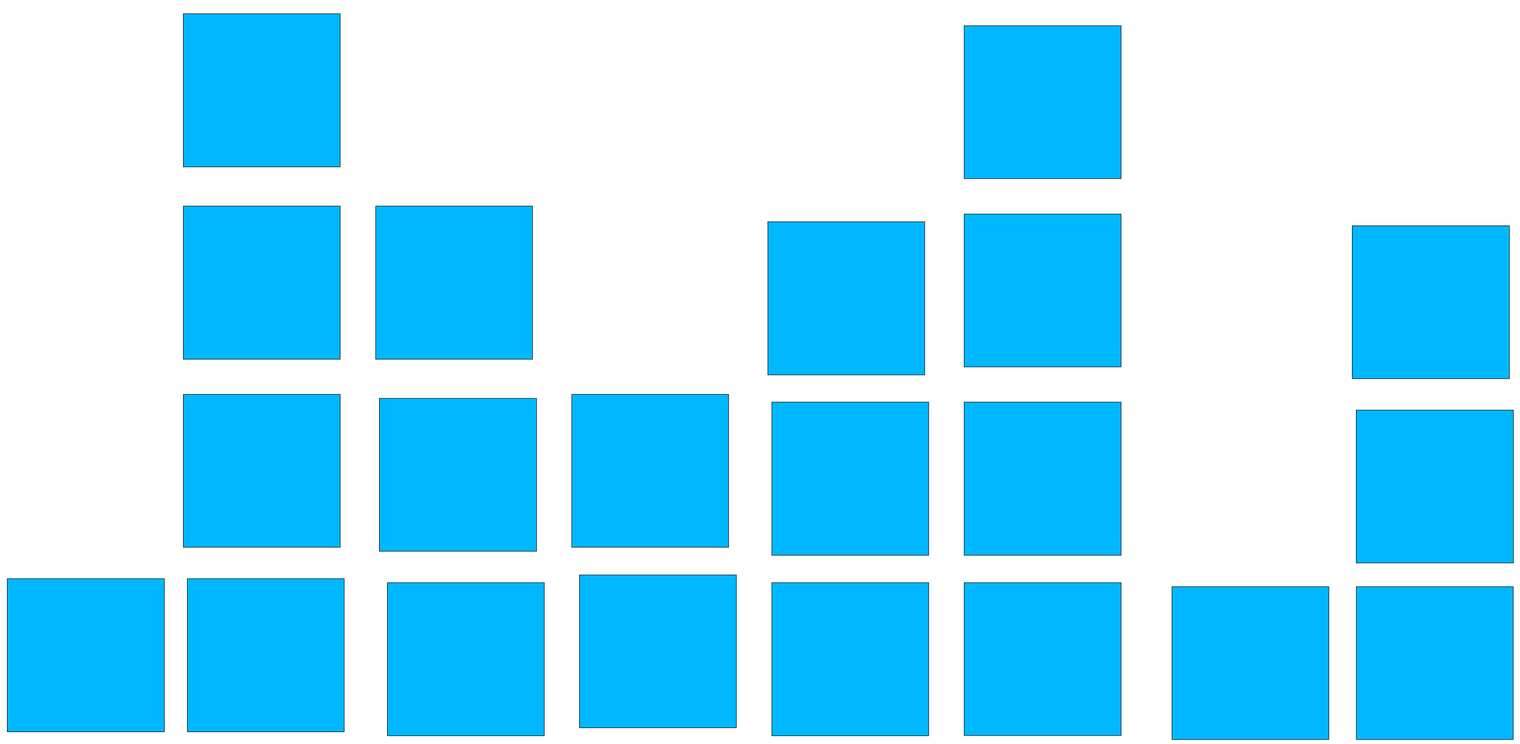
Sketch of Proof of the Kepler Conjecture

- How to reduce from an infinite number of variables to a finite (~150) number of variables.
- Nonlinear optimization problems are in general NP-hard. What special structures permit a solution to this non-linear optimization problem?

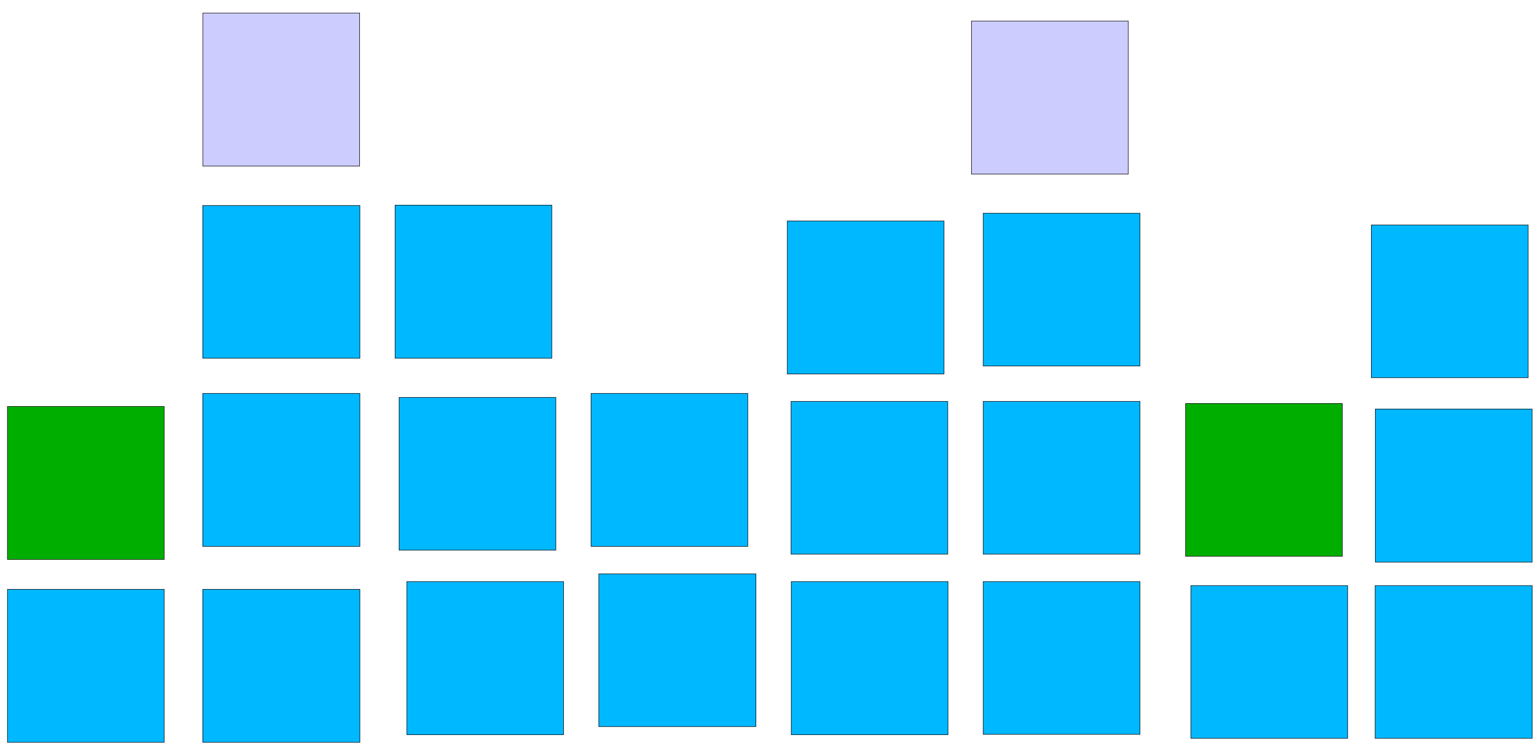
How to reduce to a finite number of variables

- A density is a limit of an average over larger and larger regions of space.
- How can infinite averages be reduced to finite averages?

An infinite brick wall ----



Reassigning the bricks in the wall



Analogy

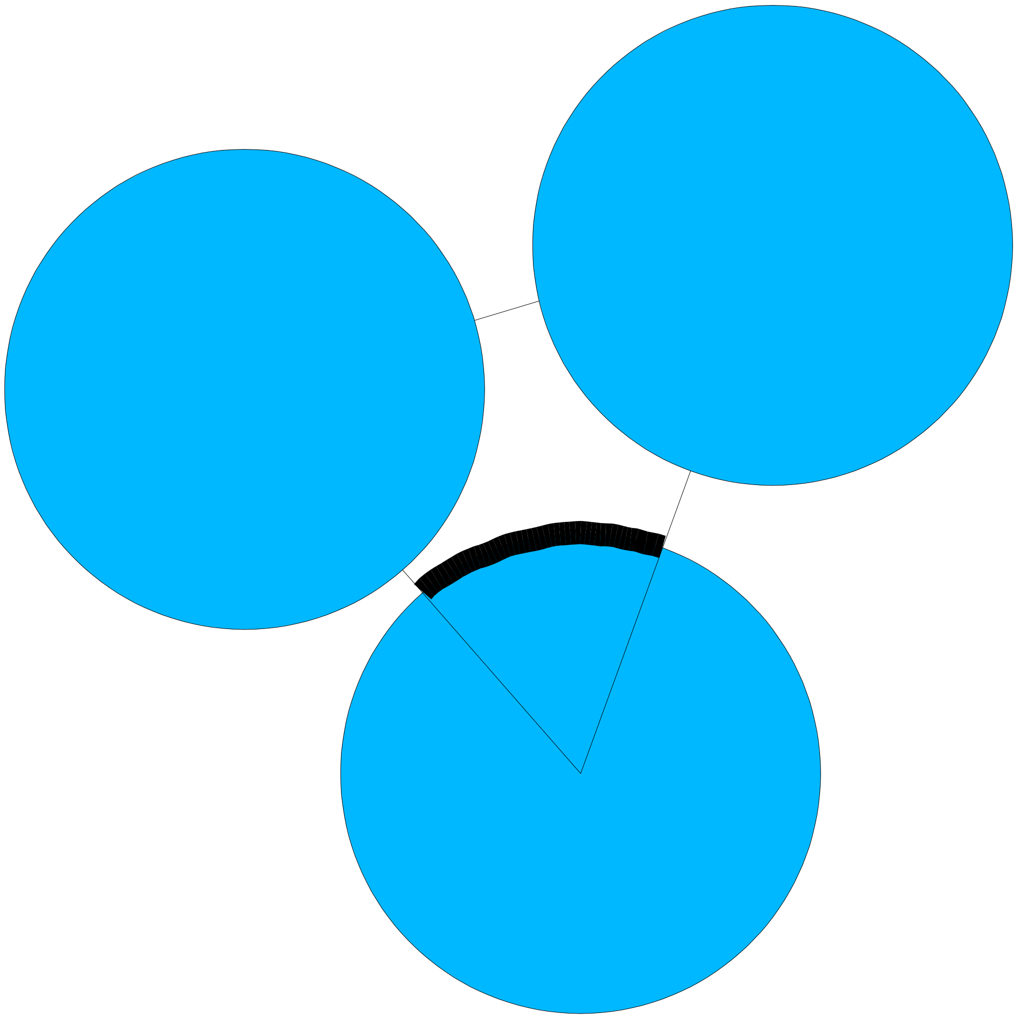
- Wall = Sphere Packing
- High parts of the wall = regions of high density
- Average ht of wall = asymptotic density
- Moving bricks about = changing geometric decomposition of space
- Highest point on the wall = local nonlinear optimization problem

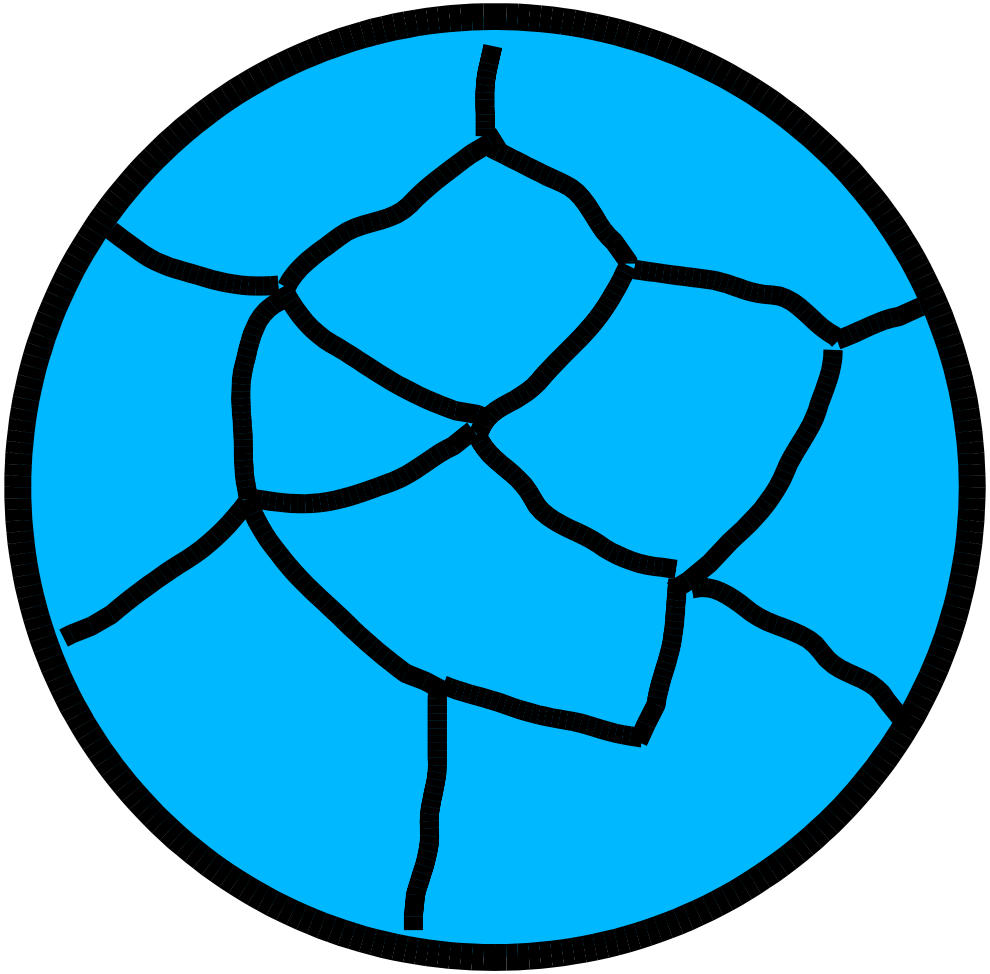
Nonlinear optimization

- The local nonlinear optimization problem is simplified considerably in two ways
  - A graph that encodes the local geometry.
  - Linear programming.

The graph associated to a local cluster of spheres.

- Let T be a cut-off parameter.
- Connect the center of every pair of spheres in the packing if the centers have distance at most T from one another.
- Fix one sphere center A, draw an arc on the sphere for every triangle (A,B,C) of centers that are connected by an edge.





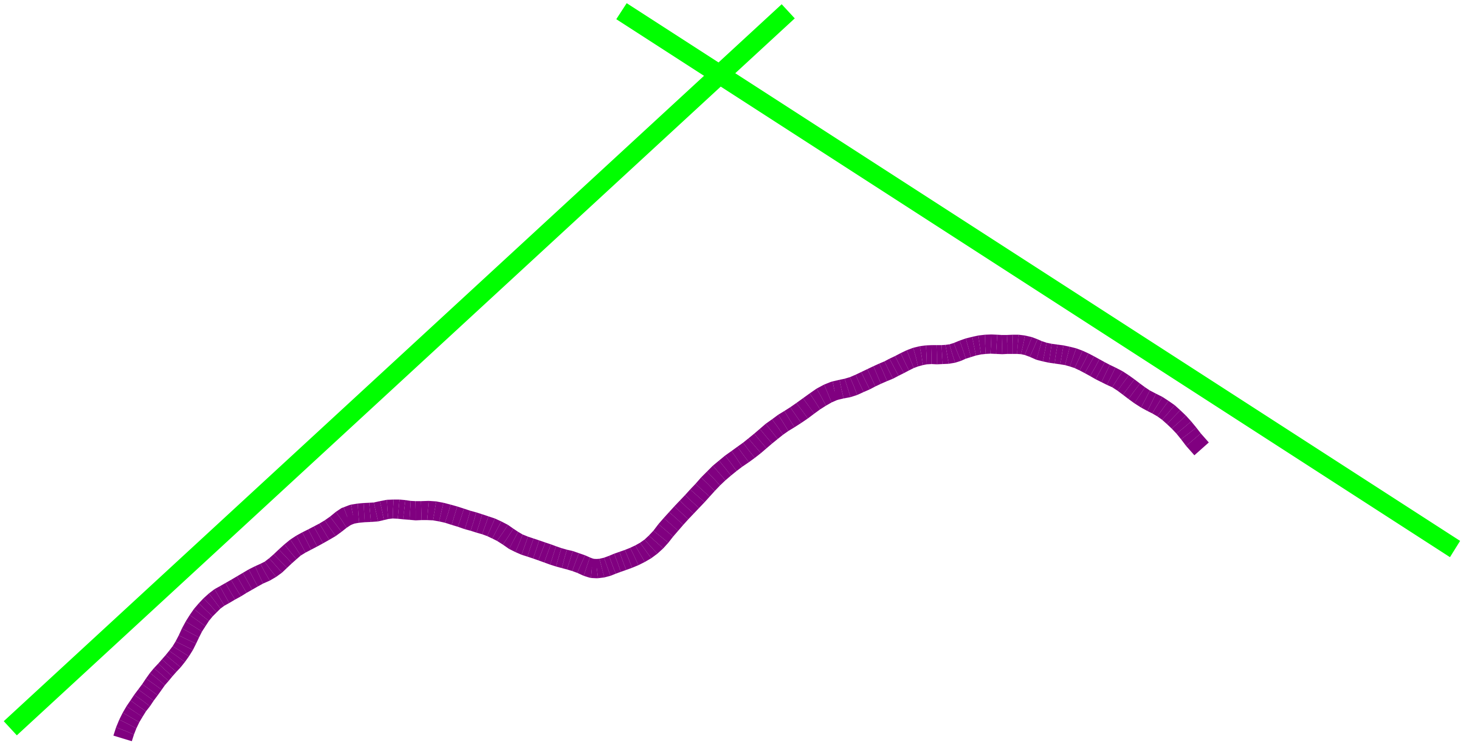
Graph Properties

- Each "face" of the planar graph gives a contribution to the nonlinear optimization problem.
- Translate properties of the nonlinear optimization problem into combinatorial properties of this planar graph.
- Classify (by computer) all planar graphs that are associated with potential counterexamples of the Kepler conjecture
- The computer yields about 5000 graphs.

Linear Programming

- Fix a planar graph and look at the nonlinear optimization problem over all local configurations attached to that planar graph.
- This tends to be a nonlinear optimization in about 30-50 variables.
- Solve by linear programming

Linear relaxation



Linear relaxation

- Each hyperplane appears in the proof in two guises:
  - Uninterpreted variables in linear programs
  - Interpreted variables in nonlinear inequalities
- Example:
  - dih + vol + y3 < 7
- To justify the use of an uninterpreted inequality as a hyperplane in a linear program, it is necessary to prove the interpreted nonlinear inequality.
- Use inequalities that involve a small number of variables that can be proved by computer.

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- There are few prerequisites:
  - Jordan curve theorem
  - Multivariable calculus.
- The rest is in HOL-light (real analysis)
- The project should take about 20 work years.
- Fortunately, several others have become involved in the project.
  - Steven Obua – Linear Programming
  - Roland Zumkeller – Nonlinear inequalities

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- The original proof is highly repetitive:
  - Similar estimates are applied to differerent cas-
es.
  - Computers were avoided in many places even when they would have been helpful.
- My immediate goal is to produce a “higher order version” of the proof – a version that is both easier to understand and easier to implement in a formal system.

References

- Google: Kepler Hales
- Google: Flyspeck Hales
- http://www.math.pitt.edu/~thales/