Semantic of Backtracking>>

(Ongoing work with S. Hayashi, T. Coquand)

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Abstract of the talk

- ***“Backtracking”*** is the possibily, for a given computation, to come back to a previous state and restart from it, forgetting everything took place after it.
- We introduce a new notion, ***“1-backtracking”***. It is a particular case of backtracking in which forgetting is ***irreversible***. If we forget a state we can never restore it back.
- We introduce a game theoretical model for 1-backtracking.

Games in Set Theory

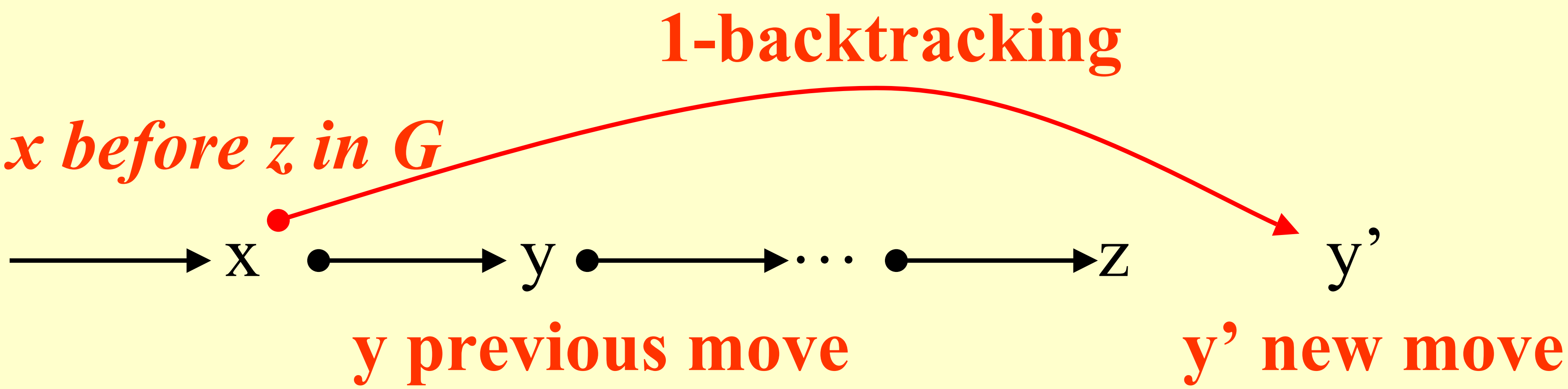
A game **G = (T, R, turn, $W_{\mathcal{E}}$, $W_{\mathcal{A}}$)** between two players, $\mathcal{E}$ (Eloise) and $\mathcal{A}$ (Abelard), consists of:

1. a tree **T** with a father/child relation **R**.
2. A map **turn**:T→{$\mathcal{E}$, $\mathcal{A}$}.
3. A partition **($W_{\mathcal{E}}$, $W_{\mathcal{A}}$)** of infinite branches of T.

A play is any finite or infinite branch of T, starting from the root of T. In each node x of the branch, the player **turn**(x) must select a child of x in T, otherwise his opponent wins. If a play continues forever, the winner is $\mathcal{E}$ if the play is in $W_{\mathcal{E}}$, and it is $\mathcal{A}$ if the play is in $W_{\mathcal{A}}$.

Games with 1-Backtracking

- Given a game G, we define a game **bck(G)**. In bck(G), a player can come back from the last position z of the play to some previous position x, ***provided x is before z in G***. Then he can replace the move y he did from x by some new move y'.



Infinite backtracking is loosing

- A player is allowed to backtrack to a given position x_i of the play only finitely many times.
- A player backtracking infinitely many times to the same position x_i looses.
- If $\mathcal{E}$ backtracks infinitely many times to some x_i and $\mathcal{A}$ infinitely many times to some x_j, the looser is the player backtracking infinitely many times to a position with smaller index.

Defining bck(G)

- **Positions of bck(G).** All finite successions $<s_0, \ldots, s_n>$ over G, such that: ***(i)*** s_0 initial position of G ***(ii)*** any s_{i+1} is a child in G of some s_j, with $j \leq i$, ***s_j ancestor in G of s_i***, and turn(s_j) = turn(s_i).
- For each infinite play $\sigma = <s_0, \ldots, s_n, \ldots>$ of bck(G) we define a (finite or infinite) play $\sigma^{(1)} = <t_0, \ldots, t_n, \ldots>$ of G by recursion on n:

1. t_0 = initial position of G.
2. t_{n+1} = last child of t_n in σ, provided t_n has a last child in σ. Otherwise $\sigma^{(1)}$ ends.

Defining bck(G)

- **Turn.** The player on turn on a position $<s_0, \ldots, s_n>$ of bck(G) is the player on turn on s_n.

- **Winner of a infinite play.** The winner of an infinite play σ of bck(G) is the winner of the (finite or infinite) play $\sigma^{(1)}$ of G associated to it.

1-Backtracking and Limit Computable Mathematic

- Let A be any arithmetical formula in the connectives ∃, ∨, ∀, ∧. Let $\mathcal{G}$(A) be the Tarski game associated to A. Let LCM be Hayashi's Limit Computable Mathematic (or "Arithmetic with incremental learning").
- **Theorem.** A is true in LCM if and only if $\mathcal{E}$ if has a recursive winning strategy on bck($\mathcal{G}$(A)).

1-backtracking characterizes the set of formulas we can "learn"

1-Backtracking and Recursive Degrees

- Let *O* be any recursive degree, G any game, p any player.
- **Theorem.** p has a winning strategy of degree *O*' for G if and only if p has a winning strategy of degree *O* for bck(G).
- In the definition of a winning strategy:

1-backtracking = 1 quantifier less

1- Backtracking and Excluded Middle

- Let A be any arithmetical formula in the connectives $\exists$, $\vee$, $\forall$, $\wedge$. Let $\mathcal{G}$(A) be the Tarski game associated to A. Let HA be Intuitionistic Arithmetic. Let 1-EM be Excluded Middle for degree 1 formulas.
- **Theorem.** $\mathcal{E}$ has a winning strategy for $\mathcal{G}$(A) if, and only if, $HA+\omega\text{-}rule+1\text{-}EM \vdash A$.

1-backtracking characterizes the set of intuitionistic consequences of 1-EM.

1-Backtracking and (unlimited) Backtracking

- 1-backtracking can be nested, even a transfinite number of times. For each game G and each ordinal α we can define $bck^{\alpha}(G)$.
- For any game H *with alternating players and no infinite plays*, T. Coquand defined Bck(H), the game with *unlimited* backtracking over H.
- **Theorem.** bck(H) is a subgame of Bck(H), and Bck(H) is a quotient of a subgame of $bck^{\omega_1}(H)$.

Unlimited backtracking can be obtained by iterating 1-backtracking.

References

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