Cut elimination for set theory

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Deduction modulo

Replace the axiom $\forall x\ (x + 0 = x)$ by the rule $x + 0 \longrightarrow x$

A theory: set of axioms and computation rules

$$\dfrac{\dfrac{\overline{\forall x\ x = x \vdash \forall x\ x = x}\ \text{Axiom}}{\forall x\ x = x \vdash 2 \times 2 = 4}\ \forall\text{-elim}}{\forall x\ x = x \vdash \exists y\ 2 \times y = 4}\ \exists\text{-intro}$$

Cut free proofs

In the empty theory: a cut free proof ends with an introduction rule (hence consistency, independence, disjunction, witness, ...)

Lost if we add axioms

Recovered in deduction modulo with rewrite rules only

Try to express theories

- with rewrite rules only,
- in such a way that cut elimination holds.

Crabbé's counterexample (1974)

Replace the axiom

$$x \in \cup(a) \Leftrightarrow \exists y \ (x \in y \wedge y \in a)$$

with the rewrite rule

$$x \in \cup(a) \longrightarrow \exists y \ (x \in y \wedge y \in a)$$

cut elimination holds (positive)

But not with the rule

$$x \in C \longrightarrow x \in a \wedge \neg(x \in x)$$

That is the question

Does the problem come

- from set theory *per se*
- or from the way axioms are oriented ?

More bad news

Besides the existence axioms, another axiom in set theory

Extensionality

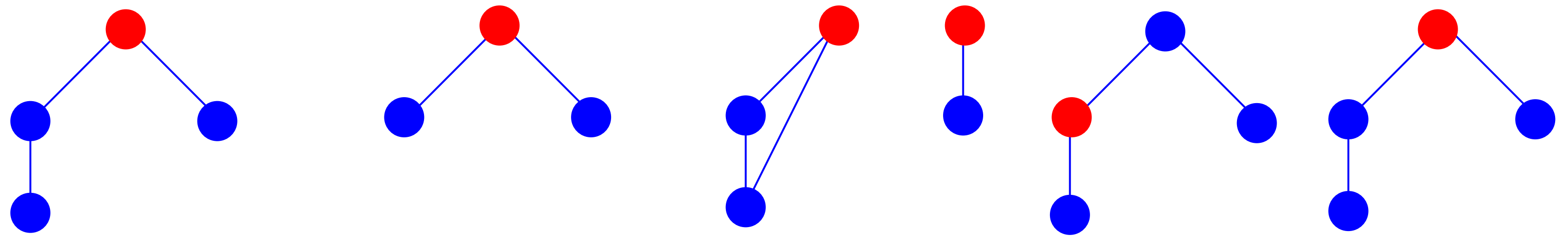
$$(\forall x\ (x \in a \Leftrightarrow x \in b)) \Rightarrow a = b$$

No clue to orient this axiom

Another presentation of set theory (Aczel, ...)

A new dogma for the XXIst century:

Mathematics should not be founded on the notion of set but on that of (directed pointed) graph



A formal theory of graphs

Two sorts (Nodes, Graphs)

Forget (temporarily) about $\in$

A new predicate: η

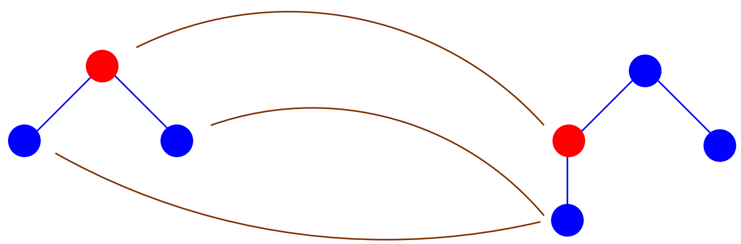
$x \ \eta_a \ y$ means "there is an edge from x to y in graph a"

More symbols: $root(a)$, a/x

Equality and membership

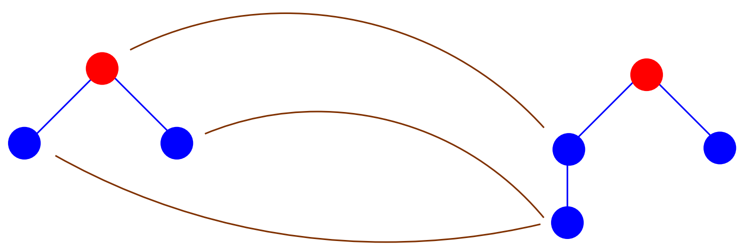
Equality as usual on nodes, but not very useful on graphs

A more useful relation: bisimilarity $\approx$



Membership derived notion: immediate subtrees up to bisimilarity

$a \in b \longrightarrow \exists x \ (x \ \eta_b \ root(b) \wedge a \approx (b/x))$



Substitutivity

$\approx$ is $\approx$-substitutive and $\in$-substitutive

We could take the quotient by $\approx$

Fixing the extensionality problem

If a and b have the same elements

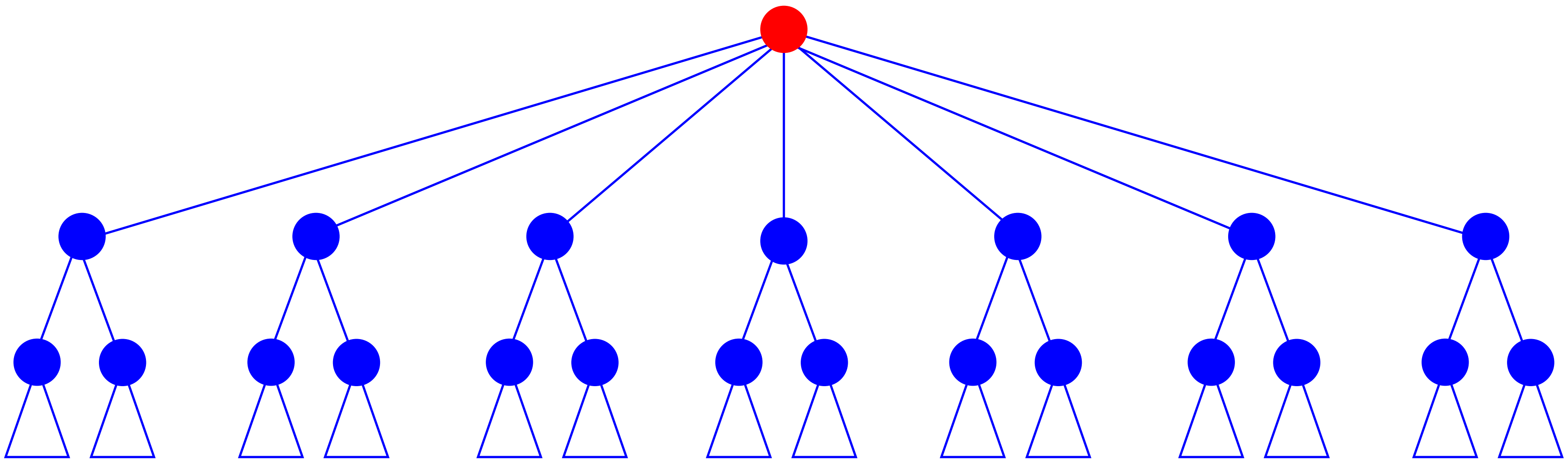
Same direct subtrees (up to $\approx$)

Then $a \approx b$

Usual approach: define equality so that it is substitutive (Leibniz equality) and then add the (embarassing) extensionality axiom

Graph theoretic approach: define equality so that it is extensional and prove it is (partially) substitutive

The usual properties of sets (e.g. $\cup$)



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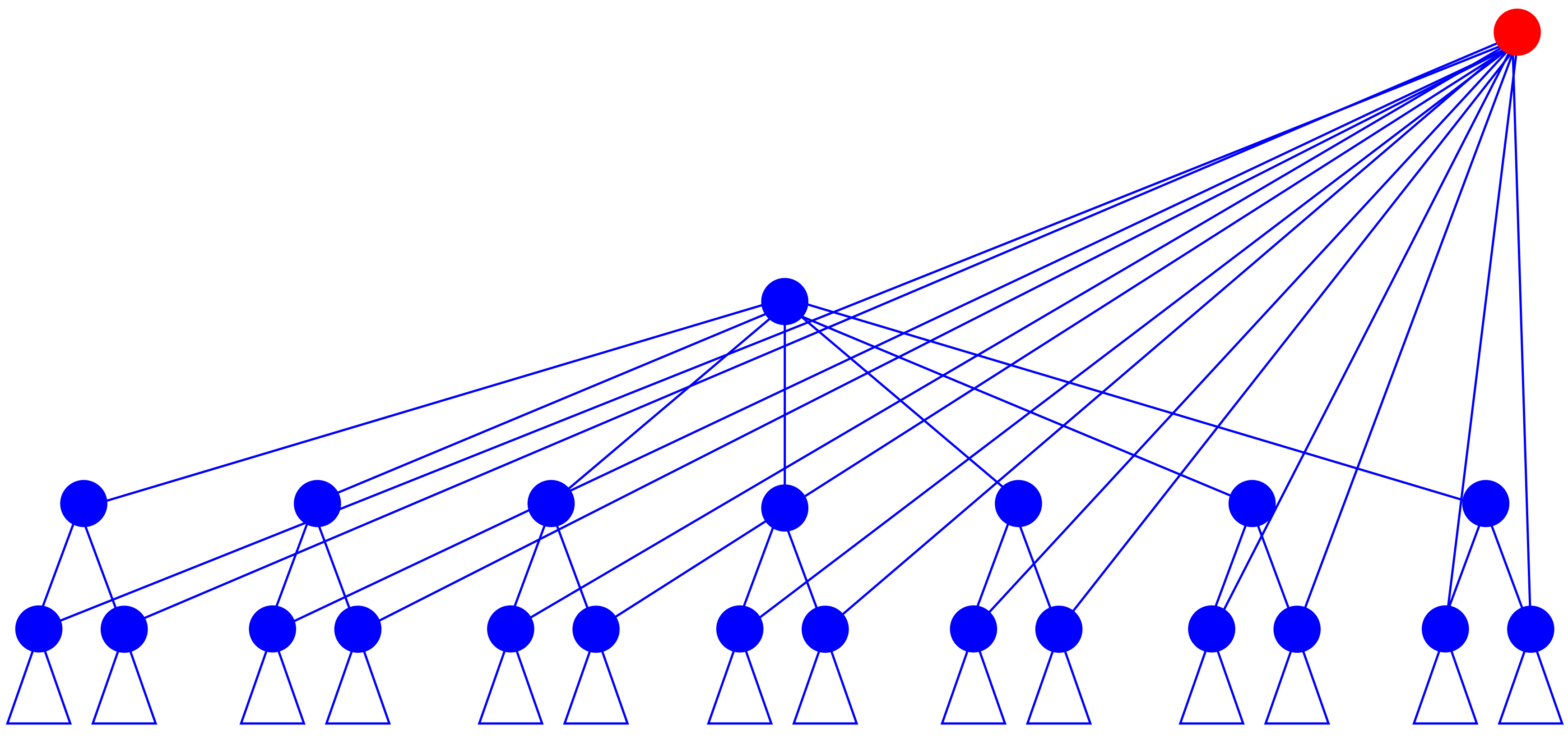

Delocalisation

Difficulty: pick a node for the root of $\cup(a)$ not in a

An non surjective injection i and an element 0 out its image

$$i'(i(x)) \longrightarrow x$$

$$I(i(x)) \longrightarrow \top \qquad\qquad I(0) \longrightarrow \bot$$

Delocalise the full graph a by i and use 0 for root of $\cup(a)$

$x\ \eta_{\cup(a)}\ x' \longrightarrow$

$$(\exists y \exists y'\ (x = i(y) \wedge x' = i(y') \wedge y\ \eta_a\ y'))$$

$$\vee(\exists y \exists z\ (x = i(y) \wedge x' = 0 \wedge y\ \eta_a\ z \wedge z\ \eta_a\ root(a)))$$

Z^{mod}

4 sorts

26 symbols (including some schemes)

30 rules (including some schemes)

- 1/3 for delocalisation (i)
- 1/3 for the constructions of set theory ($\cup$)
- 1/3 general rules ($\in$)

Equivalence

Z^{mod} is equivalent to

Zermelo's set theory + Strong extensionality + Transitive closure

Between Z and ZF + Foundation

Define two translations (one is trivial, the other interesting)

Cut elimination

To each P we associate a proposition P^* such as P provable in $\mathrm{Z}^{\mathrm{mod}}$ implies P^* provable in Z

To each P we associate a proposition $p \Vdash P$ such as p is a proof of P in $\mathrm{Z}^{\mathrm{mod}}$ implies $p \Vdash P$ provable in Z

Saturated graph: to each edge is associated a reducibility candidate

Theorem: If Z^{st} has a ω-model then all proofs in $\mathrm{Z}^{\mathrm{mod}}$ are strongly normalizing