Surreal Numbers in Coq

Types 2004 presentation

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Overview

- Introduction
- Surreal Numbers
  - interpretation
- Games
  - As Bisets
  - Sets in TT
  - Games in TT
- Order
- Summary
- Conclusions

Introduction

Surreal Numbers

- John Horton Conway, *"On Numbers and Games"*
- Class of numbers containing
  - real numbers
  - all ordinals
- Fills the "holes" between the ordinals, like $\mathbb{R}$ fills the holes between the natural numbers.
- Fills the "holes" between the reals, too.
- Equipped with totally ordered field structure

Surreal Numbers

And all this with only one set of inductive definitions!

Introduction

Surreal Numbers

And all this with only one set of inductive definitions!

Contrast with usual

$$\text{Set Theory} \longrightarrow \mathbb{N} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R}$$

construction.

Surreal Numbers

Definition

A surreal number x is a pair of arbitrary sets of surreal numbers L_x and R_x (left (resp. right) of x), with the condition:

$$\forall l \in L_x, \neg\exists r \in R_x \text{ s.t. } l \geq r$$

This x is denoted:

$$\{L_x \,|\, R_x\}$$

The class of all surreal numbers is denoted No.

Surreal Numbers

Interpretation

$x := \{L_x \,|\, R_x\}$ is the simplest number such that:

$$\forall x_l \in L_x, x > x_l$$

$$\forall x_r \in R_x, x < x_r$$

$$\underline{\qquad L_x \qquad} \; x \; \underline{\qquad R_x \qquad}$$

Surreal Numbers

Examples

$$\{\emptyset\,|\,\emptyset\} =: 0$$

$$\{\{0\}\,|\,\emptyset\} =: 1$$

$$\{\emptyset|\,\{0\}\} =: -1$$

$\{\{0\}\,|\,\{0\}\}$ Forbidden!

$\{\{1\}\,|\,\{0\}\}$ Forbidden!

$$\{\{0\}\,|\,\{1\}\} =: \tfrac{1}{2}$$

$$\{\{-1\}\,|\,\{0\}\} =: -\tfrac{1}{2}$$

$$\{\emptyset|\,\{1\}\} = 0 = \{\{-1\}\,|\,\{1\}\}$$

. . . in Coq

Games - The Need

$$x := \{L_x \,|\, R_x\}\,, \forall x_l \in L_x, \neg\exists x_r \in R_x, x_l \geq x_r$$

Mutually inductive definition of a number and of order on numbers.

Impossible in:

- Coq
- Formal mathematics as seen by Conway.

Possible in

- ALF & Co

. . . in Coq

Games

1. Games

   A game x is a pair of arbitrary sets of games.

2. Order

3. Predicate “Is a number”

Games

Bisets

Games:

$$x := \{L_x \,|\, R_x\}$$

Sets:

$$s := \{L_s\} \simeq \{L_s \,|\, \emptyset\}$$

Natural injection: collection of all sets $\longrightarrow$ collection of all games.

. . . in Coq

Sets

Two schemes:

- Subset of an existing type (universe): $s : G \longrightarrow *_p$

  Like in $\{x \in G : s(x)\}$

- Indexed by some “index set/type”: (I, f), where $f : I \longrightarrow G$

  Like in $\{f(x) : x \in I\}$

. . . in Coq

Games in Coq

```
Inductive Game : Type :=
 Game_cons : (LI,RI:Type)
             (LI -> Game) ->
             (RI -> Game) ->
             Game.
```

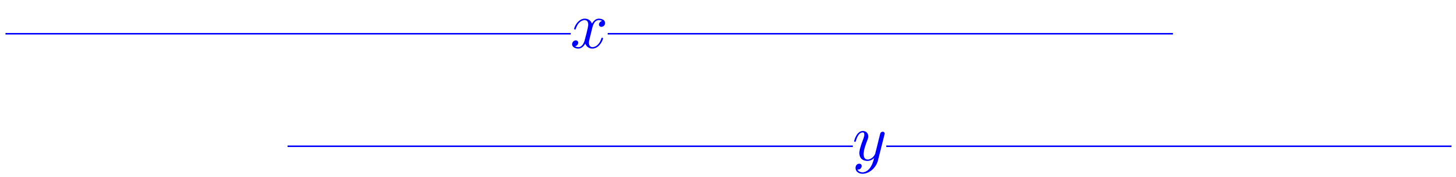
Credit: Benjamin Werner, *"Sets in Types, Types in Sets"*:

```
Inductive Ensemble : Type :=
    sup:(A:Type)(A->Ensemble)->Ensemble.
```

Surreal Numbers. . .

Order

$$x \leq y \overset{\text{def}}{\longleftrightarrow} (\forall x_l, \neg(x_l \geq y)) \wedge (\forall y_r, \neg(x \geq y_r))$$



Surreal Numbers. . .

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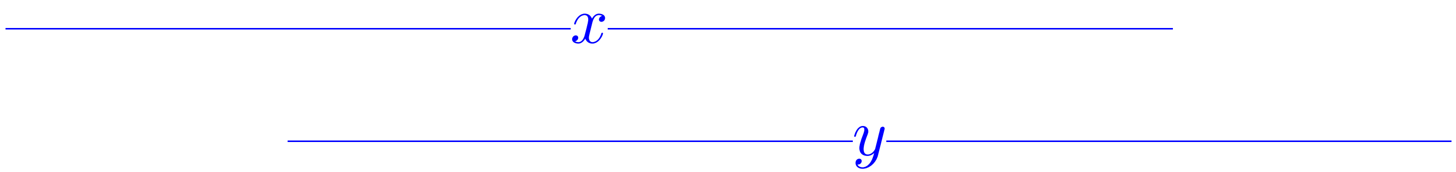

$x_l < x$

If $y \leq x_l$, then $y < x$, and $\neg(x \leq y)$.

... in Coq

Order

$$x \le y \overset{\text{def}}{\leftrightarrow} (\forall x_l, \neg(y \le x_l)) \wedge (\forall y_r, \neg(y_r \le x))$$

Cannot define $\le$ inductively from its negation! Idea from Frank Rosemeier: $\lhd \leftrightarrow \neg \ge$ and define both mutually inductive:

$$\begin{array}{rcl} x \le y & \overset{\text{def}}{\leftrightarrow} & (\forall x_l, x_l \lhd y) \wedge (\forall y_r, x \lhd y_r) \\ x \lhd y & \leftrightarrow & \neg((\forall y_l, \neg(x \le y_l)) \wedge (\forall x_r, \neg(x_r \le y))) \\ & \overset{\text{def}}{\leftrightarrow} & (\exists x_r, x_r \le y) \vee (\exists y_l, x \le y_l) \end{array}$$

Summary

Done

- Definition: games, order, “is a number”, addition, opposite, multiplication
- Properties: Commutative ordered group, distributivity, morphisms, . . .
- Nearly: Commutative ordered ring; associativity of Mult is missing.

Conclusions, Future Plans

- Many "similar" (symmetric) cases; Coq could handle that better.
- Permuting induction schemes: Easier than writing lambda-term? Automatic? (cf size-change termination criterion)

  Linked to accepting a larger class of recursive definitions.
- Defined equality: Getting much better (Claudio Sacerdoti Coen)
- More fine-grained control of reduction.
- `Destruct x`: Would want to "remember" $x = \{L_x \,|\, R_x\}$.

Closing

That's all, folks!

Questions?

Coq

Order

```
Inductive Glte : Game -> Game -> Prop :=
Glte_cons : (xLI,xRI:Type)(xLf:xLI->Game)
            (xRf:xRI->Game)(yLI,yRI:Type)
            (yLf:yLI->Game)(yRf:yRI->Game)
            [x:=(Game_cons xLI xRI xLf xRf);
             y:=(Game_cons  yLI yRI yLf yRf)]
            ((l:xLI) [xl:=(xLf l)](NGgte xl y)) ->
            ((r:yRI) (NGgte x (yRf r))) ->
            (Glte x y)
```

```
with NGgte : Game -> Game -> Prop :=
 NGgte_xr : (xLI,xRI:Type)(xLf:xLI->Game)
            (xRf:xRI->Game)(y:Game)
             [x:=(Game_cons xLI xRI xLf xRf)]
             (exT xRI [r:xRI] (Glte (xRf r) y)) ->
             (NGgte x y)
|NGgte_yl : (x:Game)(yLI,yRI:Type)(yLf:yLI->Game)
            (yRf:yRI->Game)
             [y:=(Game_cons  yLI yRI yLf yRf)]
             (exT yLI [l:yLI] (Glte x (yLf l))) ->
             (NGgte x y)
.
```

Surreal Numbers. . .

Examples - $\mathbb{N}$

$$0 := \{\emptyset \,|\, \emptyset\}$$

$$1 := \{\{0\} \,|\, \emptyset\}$$

$$2 := \{\{1\} \,|\, \emptyset\} = \{\{0, 1\} \,|\, \emptyset\}$$

$$\ldots$$

$$n + 1 := \{\{n\} \,|\, \emptyset\}$$

We now have a copy of $\mathbb{N}$ in No.

Surreal Numbers. . .

Examples - On

φ the canonical injection $On \longrightarrow No$, n an ordinal, N a set of ordinals.

$$\varphi(\text{Successor of } n) := \{\{\varphi(n)\} \mid \emptyset\} = \{\{\varphi(p), p \leq n\} \mid \emptyset\}$$

$$\varphi(\text{Limit ordinal of N}) := \{\{\varphi(n) : n \in N\} \mid \emptyset\}$$

We now have a copy of On in No.

Surreal Numbers. . .

Addition

$$x + y \stackrel{\text{def}}{=} \{x_l + y, x + y_l \,|\, x_r + y, x + y_r\}$$

Surreal Numbers...

Addition

$$x + y \stackrel{\text{def}}{=} \{x_l + y, x + y_l \,|\, x_r + y, x + y_r\}$$

$$x_l < x \rightarrow x_l + y < x + y$$

Surreal Numbers. . .

Examples

$$\frac{1}{2} := \{\{0\} \mid \{1\}\}$$

$$\frac{1}{4} := \left\{\{0\} \,\middle|\, \left\{\frac{1}{2}\right\}\right\}$$

$$\frac{3}{8} := \left\{\left\{\frac{1}{4}\right\} \,\middle|\, \left\{\frac{1}{2}\right\}\right\}$$

$$\frac{1}{3} := \left\{\left\{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{32}, \cdots\right\} \,\middle|\, \left\{\frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \cdots\right\}\right\}$$

We now have a copy of $\mathbb{R}$ in No.

Surreal Numbers...

Examples

Beyond $\mathbb{R}$...

What about $\varepsilon := \left\{ \{0\} \,\middle|\, \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \right\}$?

Surreal Numbers...

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What about $\varepsilon := \left\{ \{0\} \,\middle|\, \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \right\}$?

- $\varepsilon > 0$
- $\forall n \in \mathbb{N}, \varepsilon < \frac{1}{n}$

Surreal Numbers...

Examples

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It is an infinitesimal.

Surreal Numbers...

Examples

Beyond $\mathbb{R}$...

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It is an infinitesimal. In fact, it is $\frac{1}{\omega}$.

Surreal Numbers. . .

Examples

Beyond $\mathbb{R}$...

What about $\varepsilon := \left\{\{0\} \,\middle|\, \{\frac{1}{n} : n \in \mathbb{N}\}\right\}$?

- $\varepsilon > 0$
- $\forall n \in \mathbb{N}, \varepsilon < \frac{1}{n}$

It is an infinitesimal. In fact, it is $\frac{1}{\omega}$.

$\omega - \varepsilon, 1 - \varepsilon^2, \omega + \varepsilon, \frac{3}{7} \times \varepsilon$ all have a meaning.

Surreal Numbers...

Examples

Beyond $\mathbb{R}$... Not finished! Second-level infinitesimal:

$$\{\{0\} \mid \{\varepsilon^n : n \in \mathbb{N}\}\}$$

And you can go on... ω-th level infinitesimal, $\aleph_{13}$-th level infinitesimal, ...

... in Coq

Addition

$$x + y \overset{\text{def}}{=} \{x_l + y, x + y_l \,|\, x_r + y, x + y_r\}$$

...in Coq

Addition

$$x + y = \{x_l + y, x + y_l \,|\, x_r + y, x + y_r\}$$

$$= \{x_l + y, (+_x\; y_l) \,|\, x_r + y, (+_x\; y_r)\}$$

where $\forall z, (+_x\; z) \equiv (x + z)$

$$\stackrel{\text{def}}{=} \{x_l + y, (+_x\; y_l) \,|\, x_r + y, (+_x\; y_r)\}$$

where

$$+_x \stackrel{\text{def}}{=} \lambda z.\, \{x_l + z, (+_x\; z_l) \,|\, x_r + z, (+_x\; z_r)\}$$

. . . in Coq

Addition

$$x + y \stackrel{\text{def}}{=} \{x_l + y, (+_x\, y_l) \,|\, x_r + y, (+_x\, y_r)\}$$

$$\text{where } +_x \stackrel{\text{def}}{=} \lambda z.\, \{x_l + z, (+_x\, z_l) \,|\, x_r + z, (+_x\, z_r)\}$$

- This construction “obviously” terminates.
- But breaks the symmetry of addition: Why $+_x$ and not $+_y$?

Credits: Classical Ackerman Function example (and encouragement from Benjamin Werner)

Conclusion

Done:

- $\leq$ is a (pre-)order
- Commutative additive group
- Multiplication
  - 0 absorbent
  - 1 neutral
  - Commutativity, associativity
  - $(-x)y = -(xy) = x(-y)$
  - Distributivity

Conclusion

To do:

- Multiplication
  - Compatibility with order
  - Division
- Transitivity of order: Constructive?
- Normal form
- Irreducible Numbers
- Algebra and Analysis
- Integers, continued fractions, ...

Conclusion

Coq flaws and limitations

- Lack of inductive-recursive definitions
- Forces the mathematician to think constructively in hir structures definitions
- Throws away information
- Uncontrolled calculus (only normalisation)
- No good way to treat similar cases
- No good way to invoke intricate induction schemes
- The tools don't deal gracefully with a defined equality

Conclusion

- Formalising in Coq takes too much time.
- Some proofs in ONAG misleadingly simple.

  Formalising in Coq shows this, it is thus relevant.
- Proof assistants are tools. Document in user-oriented way. Specialised documentation for various users.
- Per-result dependency tracking

THE END

$\leq$ is an order

Summary

$\leq$ is an order on Games (pre-order on representants):

- reflexive
- transitive

Reflexivity

$$
\begin{array}{ll}
 & x \leq x \\
\leftrightarrow & \quad \{\text{Def}\} \\
 & \forall x_l, x_l \lhd x \wedge \forall x_r, x \lhd x_r \\
\leftarrow & \quad \{\text{Def}\} \\
 & \forall x_l \exists x_l', x_l \leq x_l' \wedge \forall x_r \exists x_r', x_r \leq x_r \\
\leftarrow & \quad \{\exists\text{-intro}\} \\
 & \forall x_l, x_l \leq x_l \wedge \forall x_r, x_r \leq x_r \\
\leftarrow & \quad \{\text{IH}\} \\
 & \top
\end{array}
$$

Transitivity

$$x \leq y \to y \leq z \to x \leq z$$

Here's how ONAG does it:

Since $x \leq y$, we cannot have $x_l \geq y$, and so by induction we cannot have $x_l \geq z$. Similarly we cannot have $x \geq z_r$, and so we must have $x \leq z$.

Indeed, $x \leq z \leftrightarrow \neg(x_l \geq z) \land \neg(x \geq z_r)$

Transitivity

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Who is convinced?

Transitivity

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Indeed, $x \leq z \leftrightarrow \neg(x_l \geq z) \wedge \neg(x \geq z_r)$

Who is convinced? You should not be!

Transitivity

$$x \leq y \rightarrow y \leq z \rightarrow x \leq z$$

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Transitivity

$$x \le y \to y \le z \to x \le z$$

Here's how ONAG does it:

Since $x \le y$, we cannot have $x_l \ge y$, and so by induction we cannot have $x_l \ge z$. Similarly we cannot have $x \ge z_r$, and so we must have $x \le z$.

$$x \le z \leftrightarrow \neg(x_l \ge z) \wedge \neg(x \ge z_r)$$

thus it suffices to prove:

$$x \le y \to y \le z \to \neg(x_l \ge z)$$

$$x \le y \to y \le z \to \neg(x \ge z_r)$$

Transitivity

$$x \leq y \to y \leq z \to x \leq z$$

$$x \leq y \to y \leq z \to \neg(x_l \geq z)$$

$$x \leq y \to y \leq z \to \neg(x \geq z_r)$$

Transitivity

$$x \le y \to y \le z \to x \le z$$

$$x \le y \to y \le z \to \neg(x_l \ge z)$$

$$
\begin{array}{ll}
 & x_l \ge z \\
\to & \{\text{IH}: x_l \ge z \to z \ge y \to x_l \ge y, z \ge y\} \\
 & \{\text{IH}: y \le z \to z \le x_l \to y \le x_l, y \le z\} \\
 & x_l \ge y \\
\to & \{x \le y\} \\
 & \bot
\end{array}
$$

Transitivity

$$x \leq y \to y \leq z \to x \leq z$$

$$x \leq y \to y \leq z \to \neg(x_l \geq z)$$

$$
\begin{array}{ll}
 & x_l \geq z \\
\to & \quad \{\mathsf{IH}_x : x_l \geq z \to z \geq y \to x_l \geq y, z \geq y\} \\
 & x_l \geq y \\
\to & \quad \{x \leq y\} \\
 & \bot
\end{array}
$$

Proof of $\neg(x_l \geq z)$ uses $\mathsf{IH}_x(x, z, y)$, but proof of $\neg(x \geq z_r)$ uses $\mathsf{IH}_z(y, x, z)$

$$
\begin{aligned}
& x_l \geq z \\
\rightarrow \quad & \{\mathsf{IH}_x : x_l \geq z \rightarrow z \geq y \rightarrow x_l \geq y, z \geq y\} \\
& x_l \geq y \\
\rightarrow \quad & \{x \leq y\} \\
& \bot
\end{aligned}
$$

$$
\begin{aligned}
& x \geq z_r \\
\rightarrow \quad & \{\mathsf{IH}_z : y \geq x \rightarrow x \geq z_r \rightarrow y \geq z_r, y \geq x\} \\
& y \geq z_r \\
\rightarrow \quad & \{y \leq z\} \\
& \bot
\end{aligned}
$$

Proof of $\neg(x_l \geq z)$ uses $\mathsf{IH}_x(x, z, y)$, but proof of $\neg(x \geq z_r)$ uses $\mathsf{IH}_z(y, x, z)$

But

$$(\forall x, y, z : (\forall z, y : P(x_l, z, y)) \to (\forall x, y : P(x, y, z_r)) \to P(x, y, z))$$
$$\to (\forall x, y, z : P(x, y, z))$$

is not a valid induction scheme!

Simplified version:

$$(\forall x, y : (\forall y : P(x_l, y)) \to (\forall x : P(x, y_r)) \to P(x, y))$$
$$\to (\forall x, y : P(x, y))$$

$$(\forall x, y, z : (\forall z, y : P(x_l, z, y)) \to (\forall x, y : P(x, y, z_r)) \to P(x, y, z))$$
$$\to (\forall x, y, z : P(x, y, z))$$

is not a valid induction scheme!

Correct induction scheme:

$$(\forall x, y, z : P(y, z, x_l) \to P(z_r, x, y) \to P(x, y, z))$$
$$\to (\forall x, y, z : P(x, y, z))$$