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A minimalist FOUNDATION FOR CONSTRUCTIVE MATHEMATICS

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T satisfies proofs-as-programs
$\Rightarrow$ T consistent with AC+CT

$$(\pm\mathrm{Ext})\ AC:\ \forall x\, \exists y\, R(x,y) \to \exists f\, \forall x\, R(x, fx)$$

$$CT:\ \forall f: N \to N\ \exists e\ \forall n\, \big(T(e,n,z) \,\&\, z = fn\big)$$

but $\left.\begin{array}{l} ZF \\ ML84 \\ \text{topos th.} \end{array}\right\} + AC + CT \vdash \bot$

$$\mathrm{ExtPow} + AC \vdash EM$$

$$\begin{tikzcd}
& \text{classical set theory} & \\
\text{type theory} \arrow[ur] & & \text{topos theory} \arrow[ul] \\
& \text{mTT} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

abandon propositions-as-sets

$\exists$ via the reflection principle

$\Gamma, (\exists x \in D)\, A(x) \vdash \Delta$ iff for every $d \in D$,
$\Gamma, A(d) \vdash \Delta$

$\Gamma, (\exists x \in D)\, A(x) \vdash \Delta$ iff $\Gamma, z \in D, A(z) \vdash \Delta$
z not free in Γ, Δ

$\exists$- formation

$$\frac{\Gamma, z \in D, A(z) \vdash \Delta}{\Gamma, (\exists x \in D)\, A(x) \vdash \Delta}$$

$\exists$- implicit reflection

$$\frac{\Gamma, (\exists x \in D)\, A(x) \vdash \Delta}{\Gamma, z \in D, A(z) \vdash \Delta}$$

trivialize $\Gamma = \emptyset$ $\Delta = (\exists x \in D)\, A(x)$

compose cut $(\exists x \in D)\, A(x)$

$\exists$- axiom

$$z \in D, A(z) \vdash (\exists x \in D)\, A(x)$$

compose cut $z \in D$ $A(z)$

trivialize $\Gamma = z \in D$ $\Gamma' = A(z)$

$\exists$- explicit reflection

$$\frac{\Gamma \vdash z \in D \qquad \Gamma' \vdash A(z)}{\Gamma, \Gamma' \vdash (\exists x \in D)\, A(x)}$$

minimal Type Theory   mTT

sets: exactly as in Martin-Löf

Σ-elimination

$$\frac{\begin{matrix} [z \in (\Sigma x \in B) C(x)] & & [x \in B,\ y \in C(x)] \\ M(z)\ set & d \in (\Sigma x \in B) C(x) & m(x,y) \in M(\langle x,y \rangle) \end{matrix}}{\Sigma elim(d,m) \in M(d)}$$

propositions: intuitionistic logic, A prop

$\exists$-elimination

$$\frac{\begin{matrix} & & [x \in B,\ y \in C(x)] \\ M\ prop & d \in (\exists x \in B) C(x) & m(x,y) \in M \end{matrix}}{\exists elim(d,m) \in M}$$

link: propositions-into-sets

$$\frac{A\ prop}{A\ set}$$

compatibility:

$$
\begin{tikzcd}
\text{M-L type theory} & & \text{topos theory} \\
& \text{mTT} \arrow[ul, "\text{forget prop}"] \arrow[ur, hook, "\text{props-as-monotypes Maietti 98, 01}"'] &
\end{tikzcd}
$$

toolbox: that for M-L type th.

normalization: that for M-L type th.

type checking: open, but seems clear

realizability interpretation: open, but have an idea

other literature: Howard 69-79
Mitchell - Plotkin ?
Aczel - Gambino 02
??
...