Point-free Integration Theory without Choice

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Choice

Several proposals to avoid (countable) choice in constructive mathematics.
Richman:

- Black box interpretation of $\exists$
  - Abstract data types,
    can be modeled (impredicatively) by existential types (Mitchell, Plotkin)
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More importantly, the mathematics gets more uniform.
E.g.Richman's proof of the fundamental theorem of algebra.

Richman's challenge

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To address this I will use a pointfree approach.

Point Free Topology

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Similar approaches for integration theory

Algebraic Integration Theory

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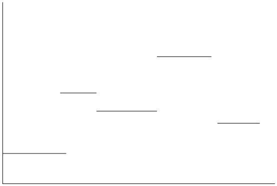
Boolean Algebra

Coquand/Palmgren:
Boolean algebra $\mathcal{A}$ of basic observables with a measure μ.
The measure defines a metric d_μ on $\mathcal{A}$.
Complete $(\mathcal{A}, d_\mu)$ to obtain a complete measure space.
[metric completion instead of σ-completion]

Boolean Algebra

Define the integral on the *simple* functions



$$\int(\sum c_i \chi_{A_i}) := \sum c_i \mu(A_i)$$

Complete wrt the norm $\int |f|$.
Riemann/Lebesgue integral is obtained from the Boolean algebra of rational intervals and $\mu((a,b)) := |b-a|$.

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Integration f-algebras

We generalize Bishop/Cheng and the Boolean algebra approach: towards integration f-algebras (formal functions)

Definition

An *f-algebra* is a vector lattice (Riesz space) which has an algebra structure compatible with the lattice space structure. That is, $fg \geq 0$, whenever $f, g \geq 0$, and if $f \wedge g = 0$, then $hf \wedge g = 0$, for all h.

One can prove that $a^2 \geq 0$, $|ab| = |a||b|$ and that if $a \wedge b = 0$, then $ab = 0$.
Example: algebras of (continuous) functions

Integration algebras

Segal's integration algebras

Definition

An *integration algebra* is a real Abelian algebra A with a linear functional I

- $I(a^2) \geq 0$ and $I(a^2) = 0$ iff $a = 0$.
- All elements are bounded: For all $b \in A$ there exists $\alpha \in \mathbf{R}$ such that $I(ba^2) \leq \alpha I(a^2)$ for all $a \in A$.

Algebra of simultaneously observable quantities.
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These can be used to give a new and constructive proof of the Peter-Weyl theorem for the representation of compact groups. (Coquand/Spitters).

Integration f-algebras

Generalizing Bishop's definition.

Definition

An *integral* I on an f-algebra is a positive linear functional I on L such that $I(a \wedge n) \to I(a)$ and $I(a \wedge 1/n) \to 0$ when n tends to ∞. An f-algebra with an integral we call an integration f-algebra.

Every integration f-algebra with a strong unit is an integration algebra.

Examples

Simple functions on a Boolean algebra, $(C(X), \int)$, $(L_1, \int)$,...

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But we need a substitute of the profile theorem.
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We use Stone representation as a substitute.

Stone representation

Stone representation theorem:
'Every f-algebra can be embedded in a space of continuous functions.'
Used in two ways:

- Substitute for the profile theorem
- Towards spectral theorem

Stone representation

Three interpretations of this space.

1. as in ordinary space, for which we need classical logic and the axiom of choice (Stone);
2. as a point-free space, which can be done constructively without the axiom of countable choice (Coquand);
3. as a compact metric space, which can be done in Bishop's constructive mathematics when R is a separable ring of normable operators on a separable Hilbert space (Bishop).

We will later see how 3 follows from 2.

Applying Stone

Let (A, I) be an integration f-algebra
By Coquand's Stone representation theorem:
A is may be embedded into a formal space of continuous functions $C(X)$ and I can be extended to $C(X)$.

Bishop's spectral theorem

Bishop's spectral theorem defines an embedding of a space of bounded integrable functions into a space of operators.
The Stone representation theorem can be used to define such an embedding for continuous functions.
Stone's theorem can be used to prove Bishop's spectral theorem.

Bishop's spectral theorem

The proof is based on a simple observation:

Theorem

Let (A, I) be an f-algebra and an integration algebra. Suppose that $I(ab^2) \geq 0$, for all $b \in A$. Then a is positive in the f-algebra A.

Consequently, $(\mathcal{A}, I)$ is an integration f-algebra.
Bishop's spectral theorem now follows from Stone's theorem, where the previous theorem is used to show that the bound is preserved.

Summary

- Observational mathematics
  - Topology
  - Measure theory
- Integration f-algebras (meeting Richman's challenge).
  - 'functions' instead of 'opens'
  - Most of Bishop's results can be generalized to this setting!
- New (easier) proof of Bishop's spectral theorem using Coquand's Stone representation theorem (formal topology)